Bounding the number of $\mathbb{F}_q$-points of hypersurfaces in (weighted) projective spaces

One of Yves's shared passions

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Frontiers in **al**gebraic **v**arieties over f**i**ni**t**e fi**e**lds
31st of August, 2026

Definition

The *weighted projective space* over $\overline{\mathbb{F}}_q$ with weights $w = (w_0, \ldots, w_m) \in \mathbb{N}_{\geqslant 1}^{m+1}$ is

$$\mathbb{P}(w) = \left(\overline{\mathbb{F}}_q^{m+1} \backslash \{0\}\right) / \sim$$

where $(x_0, \ldots, x_m) \sim (\lambda^{w_0} x_0, \ldots, \lambda^{w_m} x_m)$ for every $\lambda \in \overline{\mathbb{F}}_q^*$.

$\mathbb{P}^m \stackrel{\text{def}}{=} \mathbb{P}(1, 1, \ldots, 1)$ is the *classical* projective space.

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A point in $\mathbb{P}(w)$ is an equivalence class $[x_0 : \cdots : x_m]$ for $\sim$.
A point $[x_0 : \cdots : x_m] \in \mathbb{P}(w)$ is $\mathbb{F}_q$-rational if $(x_0^q, \ldots, x_m^q) \sim (x_0, \ldots, x_m)$. We write $\mathbb{P}(w)(\mathbb{F}_q)$ for the set of $\mathbb{F}_q$-(rational) points.

Theorem

An $\mathbb{F}_q$-point admits $q-1$ representatives in $\mathbb{F}_q^{m+1}$. $\Rightarrow p_m \overset{\text{def}}{=} |\mathbb{P}(w)(\mathbb{F}_q)| = \frac{q^{m+1}-1}{q-1}$.

In this talk, we interest in upper bounds for the number of $\mathbb{F}_q$-points at which a homogeneous polynomial of degree d vanishes in several WPS.

Part I: surfaces in $\mathbb{P}^3$

Aubry & Voloch - *Rational points on smooth surfaces in $\mathbb{P}^3$ over finite fields* (2026).

Part II: hypersurfaces in WPS

Aubry, Castryck, Ghorpade, Lachaud, O'Sullivan & Ram - *Hypersurfaces in weighted projective spaces over finite fields with applications to coding theory* (2017).

Aubry & Perret - *Maximal number of rational points on hypersurfaces in weighted projective spaces over finite fields* (2025).

N. & San José - *Maximum number of zeroes of polynomials on weighted projective spaces over a finite field* (2026).

Part I

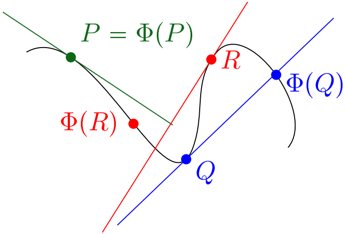
Bounds for surfaces in $\mathbb{P}^3$

with a detour throught curves in $\mathbb{P}^2$

Stöhr & Voloch - *Weierstrass points and curves over finite fields* (1986).

Take $\mathcal{C}$ a *plane* curve of degree d defined by $f(X_0, X_1, X_2) = 0$ over $\mathbb{F}_q$ with $p = \text{char}(\mathbb{F}_q) \neq 2$. Write Φ for the q–Frobenius morphism.

$$\begin{array}{rl} \mathcal{C}(\mathbb{F}_q) = & \{P \in \mathcal{C} \mid \Phi(P) = P\} \\ & \quad\quad\quad \cap\!\!| \\ & \{P \in \mathcal{C} \mid \Phi(P) \in T_P\mathcal{C}\} \overset{\text{def}}{=} \mathcal{Z} \end{array}$$



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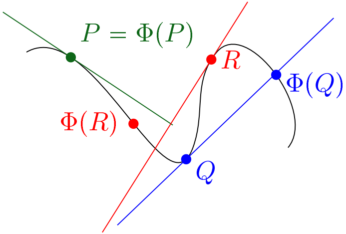

where $g = X_0^q \partial_0 f + X_1^q \partial_1 f + X_2^q \partial_2 f$.

$g(1, x_1, x_2) = (x_1^q - x_1)\partial_1 f(1, x_1, x_2) + (x_2^q - x_2)\partial_2 f(1, x_1, x_2)$

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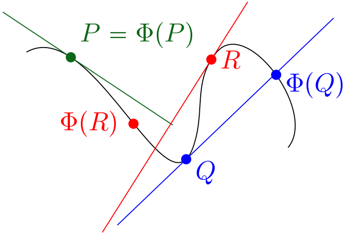

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Bézout's theorem: if $\dim \mathcal{Z} = 0$, the number of points in $\mathcal{Z}$ counted with *multiplicity* is $\deg(\mathcal{Z}) = (\deg f) \cdot (\deg g) = d(d + q - 1)$.

Multiplicity: If $P \in \mathcal{C}(\mathbb{F}_q)$, then $m_P(\mathcal{Z}) \geqslant 2$.

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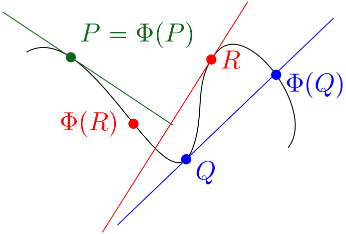

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Theorem [Stöhr–Voloch, 1986]

If $\mathcal{C}$ has at least a non–flex point ($\Rightarrow \dim \mathcal{Z} = 0$), then $|\mathcal{C}(\mathbb{F}_q)| \leqslant \frac{1}{2} d(d + q - 1)$.

Counterexamples: lines, Hermitian curve.

$\rightsquigarrow$ Stöhr–Voloch theory for curves in $\mathbb{P}^r$.

Voloch - *Surfaces in $\mathbb{P}^3$ over finite fields* (2003).

Take a degree–d surface $\mathcal{S} \subset \mathbb{P}^3$ defined by $f(X_0, X_1, X_2, X_3) = 0$ over $\mathbb{F}_q$.

Set $\mathcal{S}_1 : X_0^q \partial_0 f + X_1^q \partial_1 f + X_2^q \partial_2 f + X_3^q \partial_3 f = 0 \Rightarrow \mathcal{S} \cap \mathcal{S}_1 = \{P \in \mathcal{S} : \Phi(P) \in T_P\mathcal{S}\}$.

✔ $\mathcal{S}(\mathbb{F}_q) \subset \mathcal{S} \cap \mathcal{S}_1$ ✘ $\dim(\mathcal{S} \cap \mathcal{S}_1) \geqslant 1$ (actually 1 for most of the surfaces).

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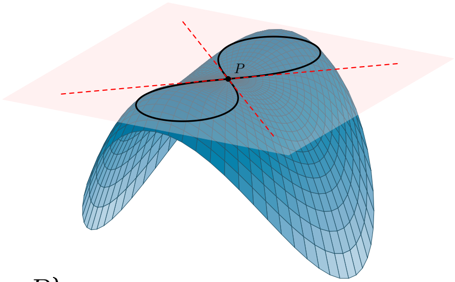
$T_P\mathcal{S}$ contains (at most) 2 asymptotic lines that meet the curve $\mathcal{S} \cap T_P\mathcal{S}$ at P with multiplicity $\geqslant 3$:

$$\sum_{0 \leqslant i \leqslant 3} \partial_i f(P) X_i = \sum_{0 \leqslant i \leqslant j \leqslant 3} \partial_{i,j} f(P) X_i X_j = 0.$$

Tikz picture generated by Claude Opus 5 (I'm ashamed) →



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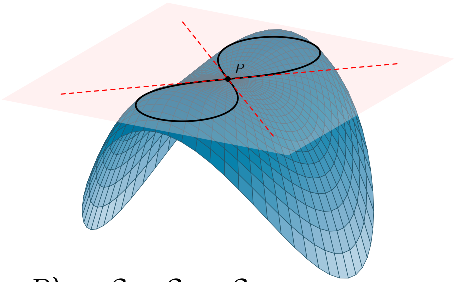

Set $\mathcal{S}_2 : \sum_{0 \leqslant i \leqslant j \leqslant 3} \partial_{i,j} f X_i^q X_j^q = 0$.

$$\mathcal{S}(\mathbb{F}_q) \subseteq \{P \in \mathcal{S} : \Phi(P) \text{ lies on the asymptotic lines at } P\} = \mathcal{S} \cap \mathcal{S}_1 \cap \mathcal{S}_2 = \underbrace{\Gamma}_{\text{0-dim.}} \cup \underbrace{L_1 \cup \cdots \cup L_m}_{\mathbb{F}_q\text{-lines on } \mathcal{S}}$$

$$\underset{\mathcal{S} \,:\, f(X_0,X_1,X_2,X_3) = 0}{d} \qquad \underset{\mathcal{S}_1 \,:\, \sum_{0\leqslant i\leqslant 3} X_i^q \partial_i f = 0}{d_1 = d+q-1} \qquad \underset{\mathcal{S}_2 \,:\, \sum_{0\leqslant i\leqslant j\leqslant 3} X_i^q X_j^q \partial_{i,j} f = 0}{d_2 = d+2q-2}$$

We have $\mathcal{S}(\mathbb{F}_q) \subset \mathcal{S} \cap \mathcal{S}_1 \cap \mathcal{S}_2 = \underset{\text{0-dim.}}{\Gamma} \cup \underset{\mathbb{F}_q\text{-lines on } \mathcal{S}}{L_1 \cup \cdots \cup L_m}$.

$\boxed{m \leqslant 11d^2 - 30d + 18 \text{ if } \mathcal{S} \text{ smooth and } d < p.}$

$$\overset{d}{\mathcal{S} : f(X_0, X_1, X_2, X_3) = 0} \qquad \overset{d_1 = d + q - 1}{\mathcal{S}_1 : \sum_{0 \leqslant i \leqslant 3} X_i^q \partial_i f = 0} \qquad \overset{d_2 = d + 2q - 2}{\mathcal{S}_2 : \sum_{0 \leqslant i \leqslant j \leqslant 3} X_i^q X_j^q \partial_{i,j} f = 0}$$

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Theorem [Voloch, 2003]

Multiplicity: If $P \in \mathcal{S}(\mathbb{F}_q)$, then $m_P(\mathcal{S} \cap \mathcal{S}_1 \cap \mathcal{S}_2) \geqslant 6$.

$$|\mathcal{S}(\mathbb{F}_q)| \leqslant \frac{d(d+q-1)(d+2q-2)}{6} + m(q+1)$$

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For $i \in \{1, 2\}$, write $\mathcal{S} \cap \mathcal{S}_i = \mathcal{C}_i \cup L_1 \cup \cdots \cup L_m$. Then $\Gamma = \mathcal{C}_1 \cap \mathcal{C}_2$.

If every L_i appears with multiplicity 1, $\mathcal{C}_i = d_i \underbrace{H}_{\text{hyperplane section}} - L_1 - \cdots - L_m \in \mathrm{Pic}(\mathcal{S})$.

$$|\Gamma| = \mathcal{C}_1 \cdot \mathcal{C}_2 = d_1 d_2 H^2 - (d_1 + d_2)(L_1 + \cdots + L_m) \cdot H + (L_1 + \cdots + L_m)^2$$

$$= d(d+q-1)(d+2q-2) - (2d+3q-3)\underbrace{m}_{L_i \cdot H = 1} + \underbrace{(L_1 + \cdots + L_m)^2}_{\leqslant m(m+1-d)}$$

Theorem [Aubry & Voloch, 2026]

If $\mathcal{S}$ is smooth of degree $2 < d < p$ and contains m $\mathbb{F}_p$-lines, then

$$|\mathcal{S}(\mathbb{F}_p)| \leqslant \frac{d(d+p-1)(d+2p-2) - 3m(d+p-1) + m(m+1)}{6} + m(p+1)$$

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Example (does not reach the bound)

The $\mathbb{F}_p$-points of the Fermat surface $\mathcal{S} : X_0^{\frac{p-1}{2}} + X_1^{\frac{p-1}{2}} + X_2^{\frac{p-1}{2}} + X_3^{\frac{p-1}{2}} = 0$ are contained in its $3d^2 = \frac{3}{4}(p-1)^2$ $\mathbb{F}_p$-lines (i.e., $|\Gamma| = 0$), and $|\mathcal{S}(\mathbb{F}_p)| = \dfrac{3}{8}\left(p^3 - 3p^2 + 11p - 9\right)$.

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You may wonder what happens for *curves* on these surfaces...

Theorem [Berardini & N., 2022]

$\dim(\mathcal{S} \cap \mathcal{S}_1) = 1$

Let $\mathcal{S} \subset \mathbb{P}^3$ be an irreducible Frobenius classical surface of degree d. Let $\mathcal{C} \subset \mathcal{S}$ be a non–plane irreducible curve of degree $\delta \leqslant q$. Then $|\mathcal{C}(\mathbb{F}_q)| \leqslant \frac{1}{2}\delta(d+q-1)$.

Holds if $\mathcal{C}$ is Frobenius nonclassical – under some conjecture, if it is Frobenius classical.

Part II

A Serre-like bound for hypersurfaces in WPS

For $f \in \mathbb{F}_q[x_0, \ldots, x_m]$ homogeneous, $V_{\mathbb{P}^m}(f)(\mathbb{F}_q) \stackrel{\text{def}}{=} \{P \in \mathbb{P}^m(\mathbb{F}_q) : f(P) = 0\}$.

Serre's bound

If $\deg(f) = d$, then $|V_{\mathbb{P}^m}(f)(\mathbb{F}_q)| \leqslant \min\left\{p_m, dq^{m-1} + p_{m-2}\right\}$. $\quad p_m = |\mathbb{P}^m(\mathbb{F}_q)|$.

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The equality is **reached.** Write $\mathbb{F}_q = \{\alpha_1, \ldots, \alpha_q\}$.

- If $d \leqslant q$, then $f = \prod_{i=1}^{d}(x_1 - \alpha_i x_0)$ defines a union of d hyperplanes meeting at the codimension 2 subspace $x_0 = x_1 = 0$.
- If $d \geqslant q+1$, then $dq^{m-1} + p_{m-2} \geqslant p_m$ and

$$f = x_0^{d-q} \prod_{i=1}^{q}(x_1 - \alpha_i x_0)$$

vanishes at every $\mathbb{F}_q$-point of $\mathbb{P}^m$.

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vanishes at every $\mathbb{F}_q$-point of $\mathbb{P}^m$.

Question: Weighted projective spaces are generalizations of $\mathbb{P}^m$. Do we have a similar bound? If yes, is it reached?

Definition

$\mathbb{P}(w_0, \dots, w_m) = \left(\overline{\mathbb{F}}_q^{m+1} \backslash \{0\}\right) /\sim$ with $(x_0, \dots, x_m) \sim (\lambda^{w_0} x_0, \dots, \lambda^{w_m} x_m), \lambda \in \overline{\mathbb{F}}_q^*$.

Isomorphisms between WPS

$\mathbb{P}(cw_0, cw_1, \dots, cw_m) \simeq \mathbb{P}(w_0, w_1, \dots, w_m)$

Delorme: $\mathbb{P}(w_0, cw_1, \dots, cw_m) \simeq \mathbb{P}(w_0, w_1, \dots, w_m)$ if $\gcd(w_0, c) = 1$.

$\Rightarrow \mathbb{P}(w_0, w_1) \simeq \mathbb{P}(w_0, 1) \simeq \mathbb{P}^1$

W.l.o.g. $w = (w_0, \dots, w_m)$ is well-formed: $\gcd(w) = \gcd(w_0, \dots, \widehat{w_i}, \dots, w_m) = 1 \ \forall i$.

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For $\mathbb{P}(1,\ldots,1)=\mathbb{P}^m$, we can choose the representatives of $\mathbb{F}_q$-points as follows:

$$(1,x_1,\ldots,x_m),\ (0,1,x_2,\ldots,x_m),\ \ldots,\ (0,\ldots,0,1) \text{ with } x_i\in\mathbb{F}_q.$$

Same for $\mathbb{P}(w_0,\ldots,w_m)$ if $\gcd(q-1,w_i)=1$ for every $i\in\{0,\ldots,m-1\}$.

Example

In $\mathbb{P}(1,2,3)$ over $\mathbb{F}_3$, $2\cdot(0,1,1)=(0,2^2,2^3)=(0,1,2)$. So $(0,1,1)\sim(0,1,2)$.

⚠ Not every $\mathbb{F}_3$-point of the line $x_0=0$ can be written $(0,1,x_2)$ with $x_2\in\mathbb{F}_3$.

- WPS are examples of **toric varieties**.
- WPS have **finite quotient singularities**.
- **Classification of surfaces:** Any surface with $K^2 = 1$ is a weighted complete intersection of type $(6, 6)$ in $\mathbb{P}(1, 2, 2, 3, 3)$. Catanese (1970)
- **Invariants and moduli spaces:**
  - The moduli spaces of elliptic curves up to isomorphism over $\mathbb{K}$ (char$(\mathbb{K}) \notin \{2, 3\}$) is

$$\mathcal{M}_{1,1} = \left\{(a : b : \Delta) : \Delta = -16(4a^3 + 27b^2)\right\} \subset \underbrace{\mathbb{P}(2, 3, 6)}_{\simeq \mathbb{P}(1,2,3)}.$$

  - Genus-2 curves up to isomorphism over $\mathbb{K}$ ($char(\mathbb{K}) \neq 2$) are characterized by Igusa invariants $(J_2, J_4, J_6, J_{10}) \in \underbrace{\mathbb{P}(2, 4, 6, 10)}_{\simeq \mathbb{P}(1,2,3,5)}$ with $J_{10} \neq 0$.

$$\deg_w(x_0^{a_0} \cdots x_m^{a_m}) = a_0 w_0 + \cdots + a_m w_m$$

$$S = \mathbb{F}_q[x_0, \ldots, x_m] = \bigoplus_{d \geqslant 0} S_d \text{ with } S_d = \mathrm{Span}\{M \text{ monomials of } S : \deg_w(M) = d\}.$$

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Conjecture [Aubry, Castryck, Ghorpade, Lachaud, O'Sullivan & Ram, 2017]

If $w_0 = 1 \leqslant w_1 \leqslant \cdots \leqslant w_m$ and $\mathrm{lcm}(w_1, \ldots, w_m) \mid d$, then

$$\max_{f \in S_d} \left| V_{\mathbb{P}(w)}(f)(\mathbb{F}_q) \right| = \min \left\{ p_m, \frac{d}{w_1} q^{m-1} + p_{m-2} \right\}.$$

- ✔ matches Serre's bound for $w = (1, \ldots, 1)$,
- ✔ proved for $m = 2$ by Aubry et al. and refined by Çakıroğlu & Şahin,
  - Çakıroğlu & Şahin - *Algebraic Invariants of Codes on Weighted Projective Planes* (2025)
- ✔ proved for $w_1 = 1$ and $\gcd(q-1, w_i, w_j) = 1$ by Aubry & Perret (25).

Aubry & Perret's strategy (1/2)

$$\mathbb{P}(w_0, \ldots, w_m)$$

$$f(x_0, \ldots, x_m) = 0$$

$$\begin{array}{ccc} \mathbb{P}(w_0,\dots,w_{m-1},1) & & f(x_0,\dots,x_{m-1},x_m^{w_m}) = 0 \\ \downarrow \pi_m & & \\ \mathbb{P}(w_0,\dots,w_m) & & f(x_0,\dots,x_m) = 0 \end{array}$$

where $\pi_i : \left\{ \begin{array}{ccc} \mathbb{P}(w_0,\dots,w_{i-1},1,w_{i+1},\dots,w_m) & \to & \mathbb{P}(w_0,\dots,w_m) \\ [x_0 : \cdots : x_m] & \mapsto & [x_0 : \cdots : x_i^{w_i} : \cdots : x_m]\,. \end{array} \right.$

Aubry & Perret's strategy (1/2)

$$
\begin{array}{ccr}
\mathbb{P}^m & & \widetilde{f}(y_0,\dots,y_m) \overset{\text{def}}{=} f(x_0^{w_0},\dots,w_m^{w_m}) = 0 \\
\downarrow \pi_0 & & \\
\vdots & & \\
\downarrow \pi_{m-2} & & \\
\mathbb{P}(w_0,\dots,w_{m-2},1,1) & & f(x_0,\dots,x_{m-1}^{w_{m-1}},x_m^{w_m}) = 0 \\
\downarrow \pi_{m-1} & & \\
\mathbb{P}(w_0,\dots,w_{m-1},1) & & f(x_0,\dots,x_{m-1},x_m^{w_m}) = 0 \\
\downarrow \pi_m & & \\
\mathbb{P}(w_0,\dots,w_m) & & f(x_0,\dots,x_m) = 0
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where $\pi_i : \left\{ \begin{array}{ccc} \mathbb{P}\left(w_0,\dots,w_{i-1},1,w_{i+1},\dots,w_m\right) & \to & \mathbb{P}\left(w_0,\dots,w_m\right) \\ [x_0 : \cdots : x_m] & \mapsto & [x_0 : \cdots : x_i^{w_i} : \cdots : x_m]\,. \end{array} \right.$

Strategy: relate $\left|V_{\mathbb{P}(w)}(f)(\mathbb{F}_q)\right|$ with $\left|V_{\mathbb{P}^m}(\widetilde{f})(\mathbb{F}_q)\right|$ and use Serre's bound for $\mathbb{P}^m$.
$\to$ Counting $\left|\pi_i^{-1}(P)\right|$ for $P \in \mathbb{P}(w_0,\dots,w_m)(\mathbb{F}_q)$.

$$\pi_i : \begin{cases} \mathbb{P}\left(w_0, \ldots, w_{i-1}, 1, w_{i+1}, \ldots, w_m\right) & \to & \mathbb{P}\left(w_0, \ldots, w_m\right) \\ \left[x_0 : \cdots : x_m\right] & \mapsto & \left[x_0 : \cdots : x_i^{w_i} : \cdots : x_m\right]. \end{cases}$$

Every $P = [x_0 : \cdots : x_m] \in \mathbb{P}(w_0, \ldots, w_m)(\mathbb{F}_q)$ with every $x_j \in \mathbb{F}_q$ falls into exactly one of this case.

- $x_i = 0 \to \pi_i^{-1}(P) = \{[x_0 : \cdots : x_{i-1} : 0 : x_{i+1} : x_m]\}$
- $x_i = 1$
- $x_i \notin \{x^{w_i}, x \in \mathbb{F}_q\} \to \pi_i^{-1}(P) = \varnothing$

$$\pi_i : \begin{cases} \mathbb{P}(w_0, \ldots, w_{i-1}, 1, w_{i+1}, \ldots, w_m) & \to & \mathbb{P}(w_0, \ldots, w_m) \\ [x_0 : \cdots : x_m] & \mapsto & [x_0 : \cdots : x_i^{w_i} : \cdots : x_m] \end{cases}.$$

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Example

Over $\mathbb{F}_5$, consider $\pi_2 : \begin{cases} \mathbb{P}(1,1,1,4) & \to & \mathbb{P}(1,1,2,4) \\ [x_0 : x_1 : x_2 : x_3] & \mapsto & [x_0 : x_1 : x_2^2 : x_3] \end{cases}.$

A point $P = [x_0 : x_1 : 1 : x_3]$ has two preimages $[x_0 : x_1 : \pm 1 : x_3]$

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A point $P = [x_0 : x_1 : 1 : x_3]$ has two preimages $[x_0 : x_1 : \pm 1 : x_3]$... unless they are the same! For example, $[0 : 0 : 1 : x_3] = [0 : 0 : -1 : x_3]$ because $2^2 = -1$.

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- $x_i = 1$

Proposition [Aubry & Perret, 2025]

Let $\delta_P = \gcd(w_j : j \in \mathrm{Supp}(P))$ and $\delta_{i,P} = \gcd(w_j : j \in \mathrm{Supp}(P)$ and $j \neq i)$ where $\mathrm{Supp}(P) = \{j : x_j \neq 0\}$. Then

$$\left|\pi_i^{-1}(P)\right|(\mathbb{F}_q) = \frac{\gcd(w_i, (q-1) \times \delta_{i,P})}{\delta_P}.$$

✘ depends on (the support of) P

If $\gcd(w_i, w_j, q-1) = 1$ for every $j \neq i$, then $\left|\pi_i^{-1}(P)\right|(\mathbb{F}_q) = \gcd(w_i, q-1)$.

✔ does not depend on P

- $x_i \notin \{x^{w_i}, x \in \mathbb{F}_q\} \to \pi_i^{-1}(P) = \varnothing$

Conjecture

If $w_0 = 1 \leqslant w_1 \leqslant \cdots \leqslant w_m$ and $\mathrm{lcm}(w_1, \ldots, w_m) \mid d$, then

$$\max_{f \in S_d} \left| V_{\mathbb{P}(w)}(f)(\mathbb{F}_q) \right| = \min \left\{ p_m, \frac{d}{w_1} q^{m-1} + p_{m-2} \right\}.$$

- ✔ matches Serre's bound for $w = (1, \ldots, 1)$,
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Theorem [N. and San José, 2025]

If $w_0 = 1$, then $\max\limits_{f \in S_d} \left| V_{\mathbb{P}(w)}(f)(\mathbb{F}_q) \right| = \min \left\{ p_m, \left(\left\lfloor \frac{d-1}{w_1} \right\rfloor + 1 \right) q^{m-1} + p_{m-2} \right\}.$

Fix a monomial order $<$ over the graded ring $S = \mathbb{F}_q[x_0, \ldots, x_m] = \bigoplus_{e \geqslant 0} S_e$.

= total ordering on monomials compatible with multiplication

The **leading monomial** $\mathrm{LM}(f)$ of $f \in S$ is the greatest monomial appearing in f.

The **footprint** of I is $\Delta(I) = \{M \text{ monomials of } S\} \setminus \{LM(f), f \in I\}$.

Footprint bound over $\mathbb{P}(w)$

$\left|V_{\mathbb{P}(w)}(I)\right| \leqslant |\Delta_e(I)|$ for *some* $e \gg 0$ where $\Delta_e(I) = \Delta(I) \cap S_e$.

Example

$I = \langle x_0, x_1^2 \rangle \subset \mathbb{F}_q[x_0, x_1]$ and $x_0 < x_1$.

$V_{\mathbb{P}^1}(I) = \varnothing$ and $\Delta(I) = \{1, x_1\}$ so $\Delta_e(I) = \varnothing$ for $e \gg 0$.

Let I be the ideal of homogeneous polynomials vanishing on $\mathbb{P}(w)(\mathbb{F}_q)$.

For f homogeneous of degree d and *some* $e \gg 0$,
$\left|V_{\mathbb{P}(w)}(f)(\mathbb{F}_q)\right| = \left|V_{\mathbb{P}(w)}(I + \langle f \rangle)\right| \leqslant \Delta_e(I + \langle f \rangle)$

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&= |\{M \text{ mon. of } S_e\} \smallsetminus \{\mathrm{LM}(h) : h \in I+\langle f\rangle\}|\\
&\leqslant |\{M \in \Delta_e(I) : \mathrm{LM}(f) \nmid M\}|\\
&= |\Delta_e(I)| - |\{M \in \Delta_e(I) : \mathrm{LM}(f) \mid M\}|
\end{aligned}$$

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Şahin - *Computing Vanishing Ideals for Toric Codes* (preprint 2025)

✔ Binomial

✘ No known generators in general

✔ If $w_0 = 1$, then $x_0 x_i \left(x_i^{q-1} - x_0^{w_i(q-1)}\right) \in I$ for $i \geqslant 1$.

$\Rightarrow$ A monomial $x_0^{b_0}\cdots x_m^{b_m} \in \Delta(I)$ with $b_0 \neq 0$ satisfies $b_i \leqslant q-1$ for all $i \geqslant 1$.

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\end{aligned}$$

Aimed bound: $\left|V_{\mathbb{P}(w)}(f)(\mathbb{F}_q)\right| \leqslant \left(\left\lfloor \frac{d-1}{w_1}\right\rfloor + 1\right) q^{m-1} + p_{m-2} = p_m - q^{m-1}\left(q - \left\lfloor \frac{d-1}{w_1}\right\rfloor\right).$

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For f homogeneous of degree d and *some* $e \gg 0$,

$$\left|V_{\mathbb{P}(w)}(f)(\mathbb{F}_q)\right| = \left|V_{\mathbb{P}(w)}(I+\langle f\rangle)\right| \leqslant \Delta_e(I+\langle f\rangle)$$
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Proposition [N. and San José, 2025]

For *most* f, $\left|\{M \in \Delta_e(I) : \mathrm{LM}(f) \mid M \text{ and } x_0 \mid M\}\right| \geqslant q^{m-1}\left(q - \left\lfloor \frac{d-1}{w_1}\right\rfloor\right)$.

✔ Reached when $\mathrm{LM}(f) = x_0^{\star} x_1^{\left\lfloor \frac{d-1}{w_1}\right\rfloor}$.

Theorem [N. and San José, 2025]

If $w_0 = 1$, then for every homogeneous polynomial f of weighted degree d

$$\left|V_{\mathbb{P}(w_0,\dots,w_m)}(f)(\mathbb{F}_q)\right| \leqslant \min\left\{p_m, \left(\left\lfloor \frac{d-1}{w_1} \right\rfloor + 1\right) q^{m-1} + p_{m-2}\right\}.$$

✔ Reached by $f = x_0^{d-w_1\left\lfloor \frac{d-1}{w_1} \right\rfloor} \prod_{i=1}^{\left\lfloor \frac{d-1}{w_1} \right\rfloor} (x_1 - \alpha_i x_0^{w_1})$ (depends only on x_0 and x_1).

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What if $w_0 \geqslant 2$?

Corollary [N. and San José, 2025]

If $\gcd(w_0, w_i, q-1) = 1$ for every $i \geqslant 1$, then the bound holds but it is not sharp.

Example: Polynomials with maximum number of $\mathbb{F}_q$-zeroes for $w = (2,3,5)$

$$f(x_0,x_1,x_2) = \begin{cases} c x_0 x_1 (x_2 - \alpha x_0 x_1) & (d = 10), \\ c x_0 (x_2 - \alpha x_0 x_1)(x_2 - \beta x_0 x_1) \text{ or } c x_0^2 x_1 (x_2 - \alpha x_0 x_1) & (d = 12) \end{cases}$$

with $\left|V_{\mathbb{P}(2,3,5)}(f)(\mathbb{F}_q)\right| = 3q$.

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Thank you for your attention!