A bound for the number of $\mathbb{F}_q$ points on a curve embedded in the biprojective space

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Member of the **Manta** project, which members are working at INRIA Saclay–Île-de-France, Télécom ParisTech and Mathematics institute of Toulouse. The geometry team of this project studies new research directions in algebraic geometry and coding theory, e.g. codes built over higher dimensional varieties.

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A message m is sent through a noisy channel. It may be altered but we want receivers to be able to *check consistency of the delivered message*, and perhaps to *recover data* that has been determined to be corrupted.
General idea: Add some **redundancy** to a message.

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$N' = 2930115155363 = 30207372735 \times 97 + 68$ and $68 + 83 \not\equiv 0$ $[97]$
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If there are more than two errors, the message cannot be recovered any more. If $\tilde{m} = \mathbf{1}01\mathbf{1}01\mathbf{0}01$, m or 101101101 ?
Correct one error + / - Message length

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A linear code of length n, dimension k and minimum distance d is called a $[n, k, d]$-code.

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We want both κ and δ big, this is not to much redundancy and a good correcting capacity. But you can't have the best of both worlds...

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$\sup_{C\ q-\text{ary}} \{\kappa(C) \mid \delta(C) = \delta\} \geq 1 - H_q(\delta)$ where
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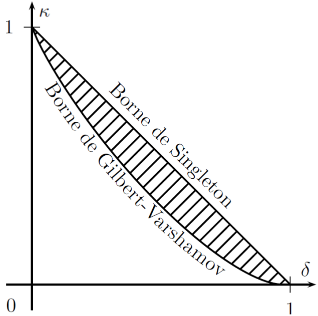

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On $\mathbb{P}^r$, fix a degree s. Take $\mathcal{F} \subset \mathbb{F}_q[X_0, X_1, \dots, X_r]$ a vector subspace of homogeneous polynomial of degree s. Fix a set of n points $\mathbb{F}_q$-rationnels $\mathcal{P} = \{P_1, \dots, P_n\} \subset \mathbb{P}^r(\mathbb{F}_q)$,

Given $f \in \mathcal{F}$ and P a point of $\mathbb{P}^r$, we define the evaluation of f at P as $f(P) := f(p_0, \dots, p_r)$, where $(p_0 : \dots : p_r)$ is the system of homogeneous coordinates of P such that the first nonzero coordinate starting from the left is set to 1, i.e. is of the form $(0 : \dots : 0 : 1 : p_i : \dots : p_n)$.

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We can define a linear code as the range of the map

$$\mathrm{ev}_s : \left\{ \begin{array}{ccc} \mathcal{F} & \to & \mathbb{F}_q^n \\ f & \mapsto & (f(P_1), \dots, f(P_n)) \end{array} \right.$$

Its length is n. Its dimension is the one of the quotient $\mathcal{F}/\ker \mathrm{ev}_s$.
Assume $\mathcal{P} = \mathbb{P}^r(\mathbb{F}_q)$. Take a codeword $\mathrm{ev}_s(f)$ and consider the hypersurface H_f defined by $f = 0$. Then

$$\omega(\mathrm{ev}_s(f)) = n - \#H_f(\mathbb{F}_q)$$

Then lowerbounding the minimum distance is equivalent to upperbound the number of $\mathbb{F}_q$-points of such hypersurfaces.

Theorem [K. Stöhr, F. Voloch]

Let $f \in \mathbb{F}_q[x,y]$ be an absolutely irreducible polynomial of degree $d \geq 2$ with coefficients in $\mathbb{F}_q$ (characteristic not 2) and denote by $\mathcal{C}$ the curve in $\mathbb{A}^2$ defined by $f = 0$. Then

$$\#\mathcal{C}(\mathbb{F}_q) \leq \frac{1}{2} d(d+q-1)$$

if at least one of the points of $\mathcal{C}$ is not an inflection point.

Idea of the proof: Consider the polynomial $h \in \mathbb{F}_q[x,y]$ defined by

$$h(x,y) = (x^q - x)f_x + (y^q - q)f_y$$

of degree $d+q-1$ and $\mathcal{H}$ the curve defined by $h = 0$.

$$\mathcal{H} \cap \mathcal{C} = \{P \in \mathcal{C} \mid \Phi(P) \in T_P\mathcal{C}\}$$

If $\mathcal{H}$ and $\mathcal{C}$ have no commun components, Bezout's Theorem gives

$$\sum_{P \in \mathcal{C} \cap \mathcal{H}} i(P;\mathcal{H},\mathcal{C}) \leq \deg f \times \deg h$$

We can prove that for any $\mathbb{F}_q$-point $P \in \mathcal{C}(\mathbb{F}_q)$ on $\mathcal{C}$, $i(P,\mathcal{H} \cap \mathcal{C}) \geq 2$.
It is true if P is singular. If P is a regular point on $\mathcal{C}$, it is enough to check that $\mathcal{H}$ and $\mathcal{C}$ have the same tangent line at P. Then

$$2\#C(\mathbb{F}_q) \leq d(d-1+q).$$

Proposition

Let $F \in \mathbb{F}_q[X_0, X_1, X_2]$ be an absolutely irreducible homogeneous polynomial of degree $d \geq 2$ with coefficients in $\mathbb{F}_q$ (characteristic not 2) and denote by $\mathcal{C}$ the curve in $\mathbb{P}^2(k)$ defined by $F = 0$. Then

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$$H = X_0^q F_{X_0} + X_1^q F_{X_1} + X_2^q F_{X_2}$$

and $\mathcal{H}$ the curve defined by $H = 0$. Using Euler Identity

$$dF = X_0 F_{X_0} + X_1 F_{X_1} + X_2 F_{X_2},$$

we can see that on each affine chart $(x_i \neq 0)$, we are back to study the intersection of f and $h(x, y) = (x^q - x) f_x + (y^q - q) f_y$.

Proposition

Let $F \in \mathbb{F}_q[X_0, X_1, Y_0, Y_1]$ be a absolutely irreducible bihomogeneous polynomial of bidegree (δ_X, δ_Y) with coefficients in the finite field $\mathbb{F}_q$ of characteristic different from 2.
Assume $\delta_X,\ \delta_Y \geq 1$.
Let $\mathcal{C}$ be the curve in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $F = 0$. Then

$$\#\mathcal{C}(\mathbb{F}_q) \leq \delta_X \delta_Y + \frac{q+1}{2}(\delta_X + \delta_Y).$$

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$$\#\mathcal{C}(\mathbb{F}_q) \leq \delta_X \delta_Y + \frac{q+1}{2}(\delta_X + \delta_Y).$$

Recall: Let $\mathcal{C}$ and $\mathcal{D}$ be two curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (δ_X, δ_Y) and (δ'_X, δ'_Y). If they have no common component, the number of intersection points, counted with multiplicity, is equal to

$$\mathcal{C} \cdot \mathcal{D} = \delta_X \delta'_Y + \delta'_X \delta_Y$$

The main idea is to *homogenize* the polynomial

$$h(x,y) = (x^q - x)f_x + (y^q - q)f_y.$$

It seems to be possible to generalize this idea to a family of surfaces, **toric surfaces**. $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are toric surfaces.

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Thank you for your attention !