Explicit construction and parameters of projective toric codes

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Inria by teleworking

Context

Take a polytope $P \subset \mathbb{R}^N$ with **integral vertices** (= convex hull of integer points)

Classical toric codes introduced by Hansen: Evaluating monomials $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_N}$ at points $(x_1, \ldots, x_N) \in (\mathbb{F}_q^*)^N$ where $m \in P \cap \mathbb{Z}^N$.
→ Well-known parameters [Hansen, Little, Soprunov-Soprunova, Ruano].

Toric codes are **algebraic-geometric codes**:
P defines a *toric variety* $\mathbf{X}_P$ and a *divisor* D.
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Aim: evaluating these fonctions on the **whole** variety.
Similar to going from Reed-Muller codes to **projective** Reed-Muller codes
Advantages:

1. length ↗, minimum distance ↗ with roughly the same dimension.
2. Strenghten the geometric interpretation

Main obstacle: Describe $\mathbf{X}_P$ and its $\mathbb{F}_q$-points to make the evaluation meaningful and *workable*

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P integral polytope of dimension N → toric variety $\mathbf{X}_P$ of dimension N
Several ways to describe $\mathbf{X}_P$: *(under some assumptions)*

- with *fans* as an abstract variety
  - $\oplus$ geometric properties
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Example: $P = \mathrm{Conv}((0,0),(1,0),(0,1),(1,1)) \subset \mathbb{R}^2$ gives $\mathbf{X}_P = \mathbb{P}^1 \times \mathbb{P}^1$:

- embedded in $\mathbb{P}^3$ by the Segre map: $(x_0, x_1, y_0, y_1) \mapsto (x_i y_j)$,
- defined as the quotient of $\left(\mathbb{A}^2 \setminus \{(0,0)\}\right)^2 \subset \mathbb{A}^4$ by the group $(\bar{\mathbb{F}}^*)^2$ via the action

  $$(\lambda,\mu)\cdot(x_0,x_1,y_0,y_1) = (\lambda x_0, \lambda x_1, \mu y_0, \mu y_1)$$

  Functions= bihomogeneous polynomials

For classical toric codes, an integral point $m \in P \cap \mathbb{Z}^N$ gives a monomial $\chi^m = X_1^{m_1} \dots X_N^{m_N}$.
In the projective case, it corresponds to a monomial $\chi^{\langle m,P\rangle} \in \mathbb{F}_\mathbf{q}[\mathbf{X_1}, \dots, \mathbf{X_r}]$.

$$L(D) = \mathrm{Span}\left(\chi^{\langle m,P\rangle} \mid m \in P \cap \mathbb{Z}^N\right)$$

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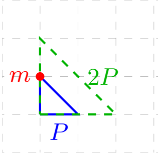

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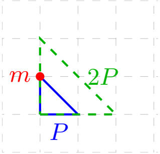

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Definition (Projective toric code)

Let P be a lattice polytope, $(\mathbf{X}_P, D)$ its corresponding toric variety and divisor. Choose a set $\mathcal{P}$ of representatives of $\mathbf{X}_P(\mathbb{F}_q)$. The *projective toric code* PC_P is defined as the image of

$$\mathrm{PC}_P = \mathrm{Span}\left\{\left(\chi^{\langle m,D\rangle}(\mathbf{x})\right)_{\mathbf{x}\in\mathcal{P}} \in \mathbb{F}_q^n,\ m \in P \cap \mathbb{Z}^N\right\}$$

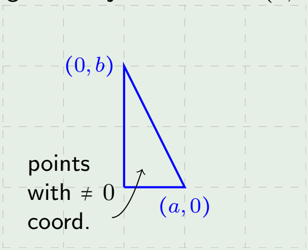
where $n = \#\mathbf{X}_P(\mathbb{F}_q)$.

The variety $\mathbf{X}_P$ is the disjoint union of tori : $\mathbf{X}_P = \bigsqcup_{Q \text{ faces of } P} \mathbb{T}_Q$

with $\mathbb{T}_Q = (\overline{\mathbb{F}_q}^*)^{\dim Q} \Rightarrow \#\mathbb{T}_Q(\mathbb{F}_q) = (q-1)^{\dim Q}$.

Examples

Weighted Projective Plane $\mathbb{P}(1,a,b)$



$\#\mathbb{P}(1,a,b)(\mathbb{F}_q) = (q-1)^2$

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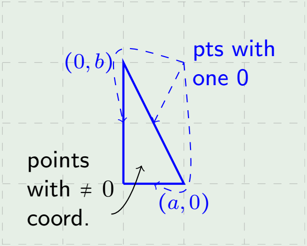

$\#\mathbb{P}(1,a,b)(\mathbb{F}_q) = (q-1)^2 + 3(q-1)$

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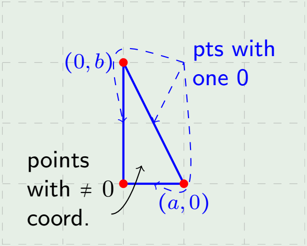

$\#\mathbb{P}(1,a,b)(\mathbb{F}_q) = (q-1)^2 + 3(q-1) + 3$

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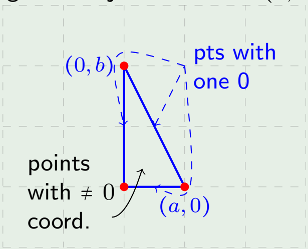

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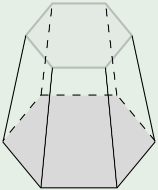
A random toric 3-fold



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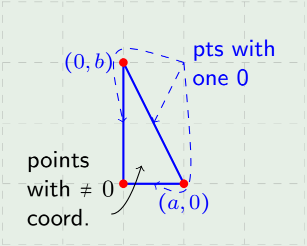

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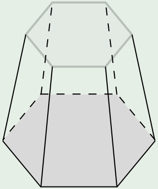

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Number of $\mathbb{F}_q$-points of $\mathbf{X}_P$

$$\#\mathbf{X}_P(\mathbb{F}_q) = (q-1)^N + \sum_{i=0}^{N-1} (\text{nb of } i\text{-dim faces}) \times (q-1)^i.$$

Evaluation

$$\mathbf{X}_P = \bigsqcup_{Q \text{ faces of } P} \mathbb{T}_Q$$

What does a codeword of PC_P look like when restricting on points of a torus $\mathbb{T}_Q$?

Recall: Integral point $m \in P \cap \mathbb{Z}^N \leftrightarrow$ Monomial $\chi^{\langle m,P\rangle} \in L(D)$

Lemma

- If $m \in Q$, $\chi^{\langle m,P\rangle}(\mathbf{x}) \neq 0 \iff \mathbf{x} \in \mathbb{T}_Q$,
- For any face Q of P, the puncturing of the code PC_P at coordinates corresponding to points of outside $\mathbb{T}_Q$ is monomially equivalent to the classical toric code C_Q.

For a face Q of P, puncturing of PC_P outside $\mathbb{T}_Q \simeq \mathrm{C}_Q$.

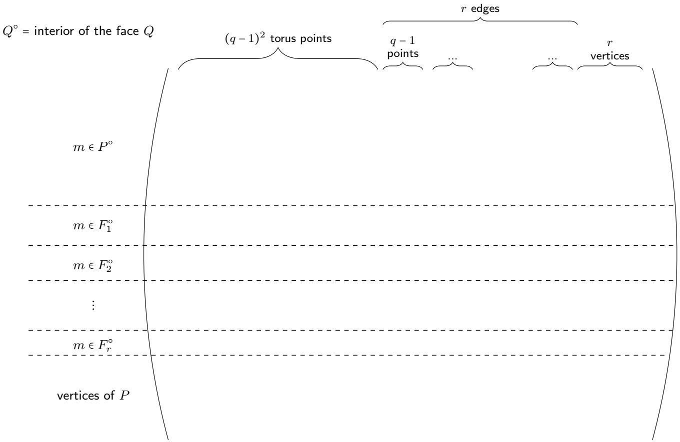

Figure: Matrix of the evaluation map associated to a polygon P $(N = 2)$

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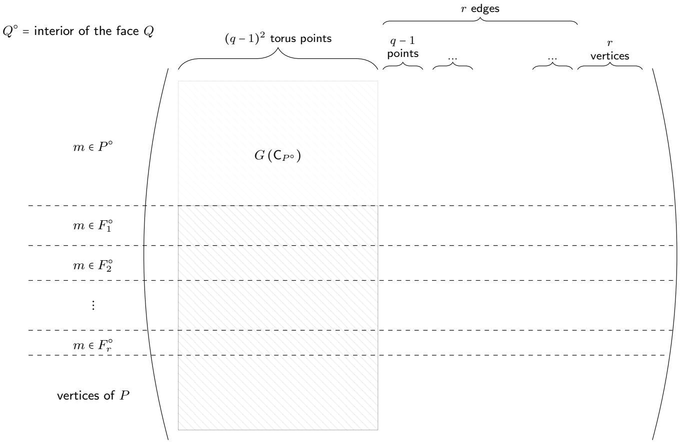

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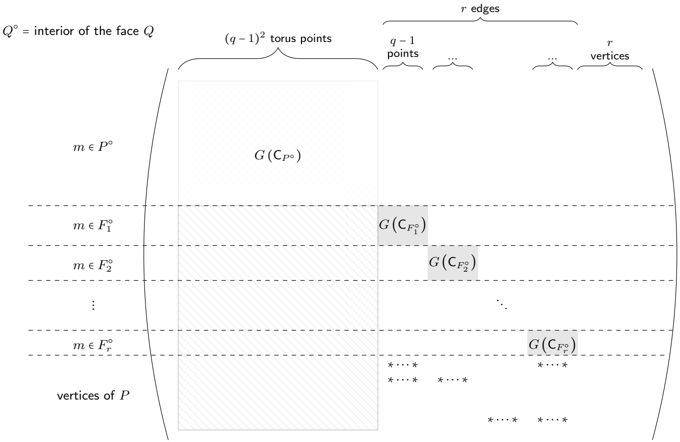

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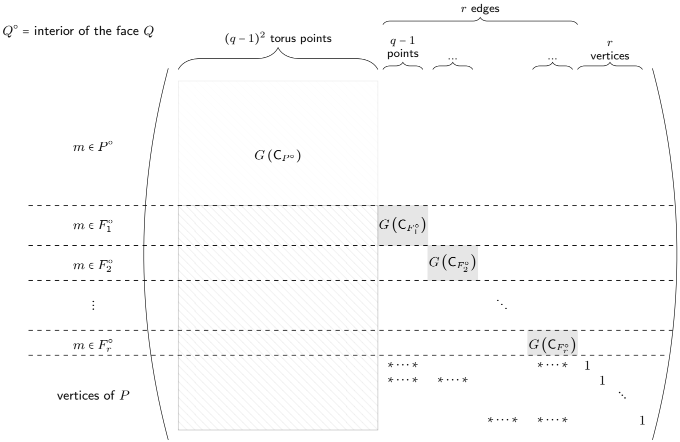

Figure: Matrix of the evaluation map associated to a polygon P ($N = 2$)

For any polytope P, there is a *generator matrix* of PC_P with such a triangular block structure.

Dimension and reduction modulo $q-1$

$$\text{Dimension of } \mathrm{PC}_P = \text{rank of the previous matrix} = \sum_Q \dim \mathrm{C}_{Q^\circ}$$

Dimension of classical toric codes

For two elements $(u,v) \in (\mathbb{Z}^N)^2$, we write $u \sim v$ if $u - v \in (q-1)\mathbb{Z}^N$.

Theorem [Ruano 07]

Let $\overline{P}$ be a set of representatives of $P \cap \mathbb{Z}^N$ under $\sim$. Then

- $\chi^m(\mathbf{t}) = \chi^{m'}(\mathbf{t})$ for every $\mathbf{t} \in (\mathbb{F}_q^*)^N \Leftrightarrow m \sim m'$,
- $\left\{(\chi^{\overline{m}}(\mathbf{t}),\ \mathbf{t} \in (\mathbb{F}_q^*)^N) \mid \overline{m} \in \overline{P}\right\}$ is a basis of C_P.

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In the projective case, the polytope P is reduced modulo $q-1$ **face by face**. On $P \cap \mathbb{Z}^N$, we write $m \sim_P m'$ if there exists a face Q of P s.t. $m, m' \in Q^\circ$ and $m - m' \in (q-1)\mathbb{Z}^N$.

Theorem [N. 20]

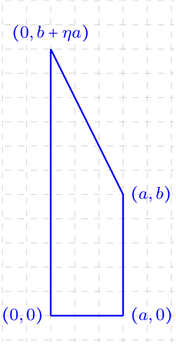
Let $\mathrm{Red}(P)$ be a set of representatives of $P \cap \mathbb{Z}^N$ modulo $\sim_P$. Then

- $\ker \mathrm{ev}_P = \mathrm{Span}\{\chi^m - \chi^{m'} : m \sim_P m'\}$,
- $\left\{\mathrm{ev}_P(\chi^{\langle \overline{m}, P\rangle}) \mid \overline{m} \in \mathrm{Red}(P)\right\}$ is a basis of PC_P.

Example of computation of the dimension of PC_P and C_P

Let a, b, $\eta \in \mathbb{N}^*$ and $P = \mathrm{Conv}((0,0),(a,0),(a,b),(0,b+\eta a))$.
→ Toric surface parametrized by the integer η called a *Hirzebruch surface* + a divisor of *bidegree* (a,b).
Let us compare the $\dim \mathsf{PC}_P$ and $\dim \mathsf{C}_P$ on $\mathbb{F}_7$ for different (a,b).
→ Reduce the interior of each face modulo $q-1=6$.

$(a,b)=(3,5)$



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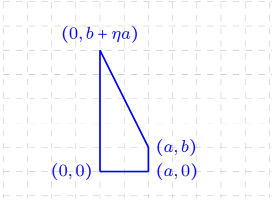

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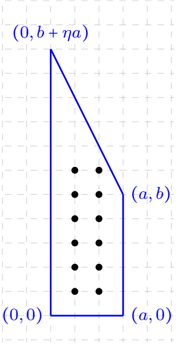

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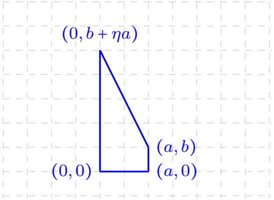

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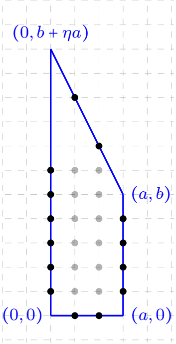

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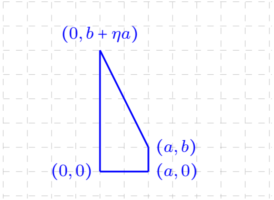

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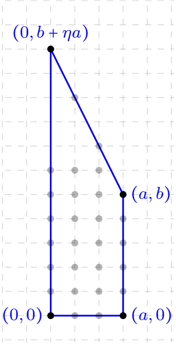

$\dim \mathsf{PC}_P = 30$

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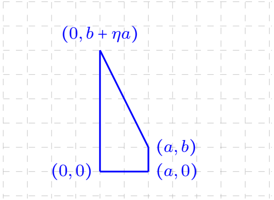

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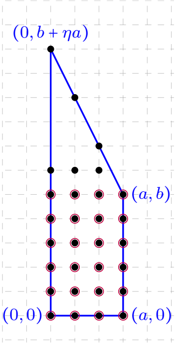

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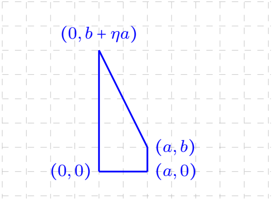

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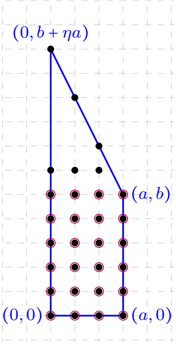

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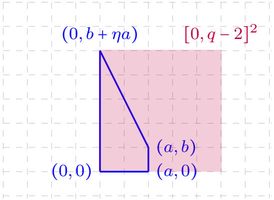
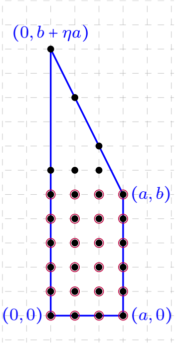


Example of computation of the dimension of PC_P and C_P

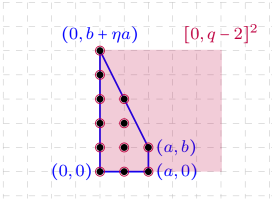
Let a, b, $\eta \in \mathbb{N}^*$ and $P = \mathrm{Conv}((0,0),(a,0),(a,b),(0,b+\eta a))$.
→ Toric surface parametrized by the integer η called a *Hirzebruch surface* + a divisor of *bidegree* (a,b).
Let us compare the $\dim \mathsf{PC}_P$ and $\dim \mathsf{C}_P$ on $\mathbb{F}_7$ for different (a,b).
→ Reduce the interior of each face modulo $q-1=6$.

$(a,b)=(3,5)$



$\dim \mathsf{PC}_P = 30 > \dim \mathsf{C}_P = 24$

$(a,b)=(2,1)$



$\dim \mathsf{PC}_P = \dim \mathsf{C}_P = \#P \cap \mathbb{Z}^2 = 12$

Lowerbound on the minimum distance on a toy example on $\mathbb{F}_4$

SECRET INGREDIENT: *Gröbner basis* of the vanishing ideal of $\mathbf{X}_P(\mathbb{F}_q)$

1. Choose a *nice* total order $<$ on $\mathbb{Z}^N$ (addition compatibility) : lexicographic
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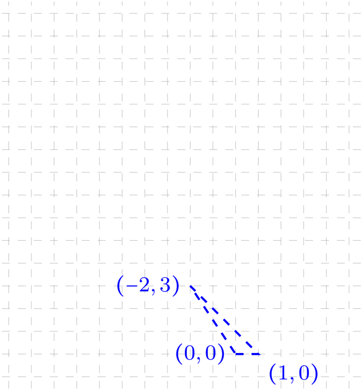

Theorem [N. 20]

$$d(\mathrm{PC}_P) \geq \min_{m \in \mathrm{Red}_<(P)} \#\left((m + P_{\mathrm{surj}} - P) \cap \mathrm{Red}_<(P_{\mathrm{surj}})\right).$$

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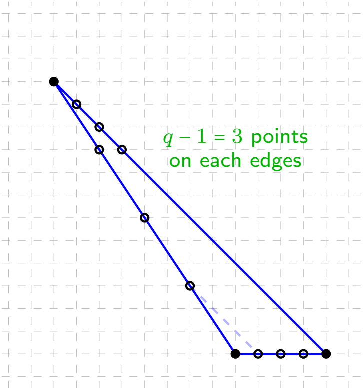

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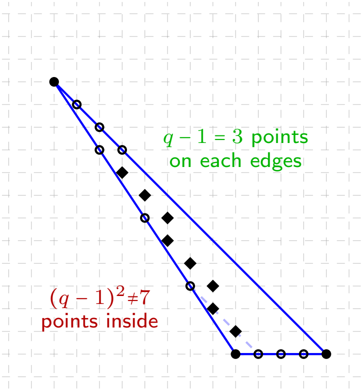

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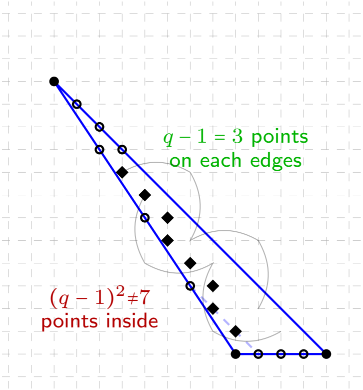

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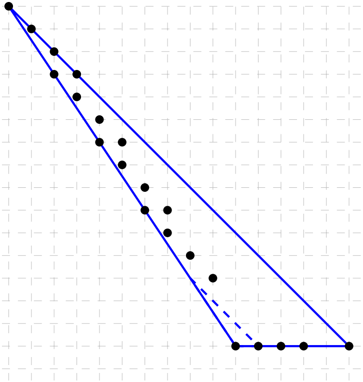

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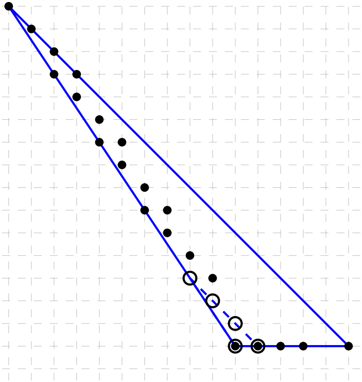

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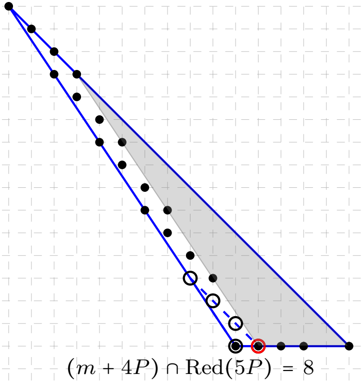

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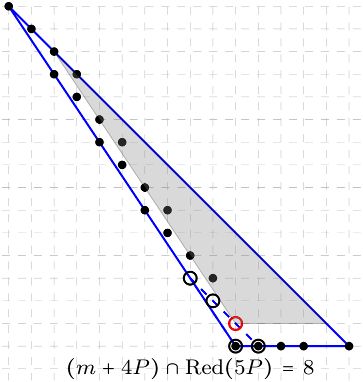
   → PC_P has type $[21, 4, 8]$



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Given a polytope P, we can

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What now?

- Investigate properties of these codes (local decodability, dual codes)
- Application to secret sharing, generalizing one based on classical toric codes by Hansen

Thank you!